SEMESTRAL EXAM - DIFFERENTIAL TOPOLOGY
MMATH II

Time : 3 hours. Max. Marks : 40

Answer all questions with proper justification(s). All questions carry equal marks.

(1) Let X be the subset of $\mathrm{Sym}(2) = \mathbb{R}^3$ defined by

$$X = \left\{ A = \begin{pmatrix} x & y \\ y & z \end{pmatrix} : \det(A) = -1,\ \mathrm{tr}(A) = 0 \right\} \subseteq \mathrm{Sym}(2).$$

Is X is a submanifold of $\mathrm{Sym}(2)$?

(2) Show that if $f : S^1 \longrightarrow \mathbb{R}$ is smooth and y a regular value of f, then $f^{-1}(y)$ consists of even number of points.

(3) Let $F : M_n(\mathbb{R}) \longrightarrow \mathbb{R}^n$ be the map that sends a matrix to the first column vector. This restricts to a smooth map

$$f = F/O(n) : O(n) \longrightarrow \mathbb{S}^{n-1}.$$

Show that f is a submersion. Is $f/SO(n)$ a submersion?

(4) Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be the map $f(x, y, z) = (xy, yz)$. Is f transverse to S^1?

(5) Let ω be the 2-form on $\mathbb{R}^3$ defined by

$$\omega = x\, dy \wedge dz - y\, dx \wedge dz + z\, dx \wedge dy.$$

Compute

$$\int_{\mathbb{S}^2} \omega.$$

Justify all your claims and give detailed computations.